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Fuzzy Quantum Logics as a Basis for Quantum Probability Theory

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Representation of an abstract quantum logic with an ordering set of states S in the form of a family $L(S)$ of fuzzy subsets of S which fulfils conditions analogous to Kolmogorovian conditions imposed on a σ-algebra of random events allows us to construct quantum probability calculus in a way completely parallel to the classical Kolmogorovian probability calculus. It is shown that the quantum probability calculus so constructed is a proper generalization of the classical Kolmogorovian one. Some indications for building a phase-space representation of quantum mechanics free of the problem of negative probabilities are given.

1. INTRODUCTION

Quantum mechanics is a probabilistic theory: in general it does not predict results of individual experiments performed on microobjects, but only probabilities of obtaining various results. These probabilities can be later compared with experimentally established relative frequencies; since during the last 70 years of development of quantum mechanics no significant discrepancy has been found between values obtained from the theory and from experiment, only a few doubt the validity and predictive power of quantum theory.

In some experiments, however, the obtained relative frequencies do not fulfil numerical constraints imposed by classical (Kolmogorovian) probability theory and the same applies to their theoretical counterparts interpreted as probabilities. Such instances, usually connected with the violation of Bell's inequalities, strongly indicate the necessity of modification of the probability calculus used in quantum mechanics.

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Actually, such a modification has already been done: In the quantum logic approach to foundations of quantum mechanics (see, e.g., Beltrametti and Cassinelli, 1981; Pták and Pulmannová, 1991) the Kolmogorovian triple $(\Omega, \mathcal{F}, P)$ consisting of a space of elementary events Ω, a Boolean σ-algebra $\mathcal{F}$ of selected subsets of Ω, and a probability measure P is replaced by a couple (L, p) consisting of a σ-orthocomplete orthomodular poset L and a probability measure (state) p defined on L. However, in the course of such a generalization (Boolean algebras are special cases of orthomodular posets) one element of Kolmogorov's (1933) original construction is lost: random events are no longer subsets of the space of elementary events. In the Hilbert space model they are represented by closed subspaces of a Hilbert space (or orthogonal projections onto such subspaces), while in an abstract model they are simply elements of an orthomodular poset. This missing element of the construction not only makes the quantum probability calculus less similar to the classical, Kolmogorovian, one, but also obscures relations between quantum mechanics and classical statistical mechanics. In the author's opinion, it also can be a source of difficulties encountered in attempts to represent quantum mechanics on a phase space.

The aim of the present paper is to show the possibility of building a quantum probability calculus on a suitably chosen families of fuzzy subsets of elementary events in a way completely analogous to the orthodox construction of Kolmogorov.

2. QUANTUM PROBABILITY CALCULUS IN THE QUANTUM LOGIC APPROACH

Following the traditional, although misleading terminology, by a *quantum logic*[2] we mean an orthomodular σ-orthocomplete orthocomplemented poset, i.e., a partially ordered set L containing the smallest element $\mathbf{0}$, the greatest element $\mathbf{1}$, and equipped with an orthocomplementation mapping $\perp$: $L \to L$ such that the following conditions are fulfilled for any $a, b, c \in L$:

(i) $(a^{\perp})^{\perp} = a$ (idempotency)

(ii) If $a \le b$, then $b^{\perp} \le a^{\perp}$ (order-reversing).

(iii) $a \wedge a^{\perp} = \mathbf{0}$ (law of contradiction), and $a \vee a^{\perp} = \mathbf{1}$ (excluded middle law), where $\wedge$ is the meet and $\vee$ is the join with respect to a given partial order in L.

[2] In the author's opinion the name *orthomodular algebra* coined by Burmeister and Mączyński (1994) is much better since it stresses the algebraic, not logical character of this object, which, moreover, describes not only quantum, but also classical systems.

(iv) If $a_i \leq a_j^{\perp}$ for $i \neq j$ (in which case we write $a_i \perp a_j$ and call these elements *orthogonal*), then the join $\vee_i a_i$ exists in L (σ-orthocompleteness condition).
(v) If $a \leq b$, then $b = a \vee (a^{\perp} \wedge b)$ (orthomodular identity).

Elements of a logic represent dichotomic observables pertaining to the physical system under study, i.e., observables that have only two possible outcomes, usually interpreted as "yes" and "no." Therefore, they also can be thought of as representing properties of a physical system (possessed or not), propositions (true or false), dichotomic tests, or yes–no questions. From the probabilistic point of view they represent random events which, when a suitable experiment is completed, are found either to have occurred or not to have occurred.

Probability measure (called also *state*) on a logic L is defined as a mapping $p\colon L \to [0, 1]$ such that:

(i) $p(\mathbf{1}) = 1$ (normalization condition).
(ii) $p(\vee_i a_i) = \Sigma_i p(a_i)$ for any sequence $\{a_i\}$ of pairwise orthogonal elements of a logic L (σ-additivity).

A set S of probability measures (states) on a logic L is called *ordering* iff $p(a) \leq p(b)$ for all $p \in S$ implies $a \leq b$.

As mentioned in the Introduction, Boolean σ-algebras of subsets of a space of elementary events Ω which form a basis for Kolmogorovian probability calculus are special cases of quantum logics in which partial order, orthocomplementation, meets, and joins are represented, respectively, by set-theoretic inclusion, complementation, unions, and intersections, **0** is represented by the empty set ∅ and **1** by the whole space Ω. Of course in this case the notion of a probability measure on a logic coincides with the traditional notion of Kolmogorovian probability measure.

The standard example of a "genuine" quantum logic (i.e., logic that is non-Boolean and can be used to describe genuine quantum systems) is a *Hilbertian quantum logic* $L(\mathcal{H})$ consisting of closed subspaces of a Hilbert space $\mathcal{H}$ used to describe a quantum system or, equivalently, orthogonal projections onto these closed subspaces. Probability measures on $L(\mathcal{H})$ are generated by density operators via the formula

$$p_{\hat{\rho}}(\hat{A}) = Tr(\hat{\rho}\hat{A}) \tag{1}$$

where $\hat{\rho}$ is a density operator representing a state of a physical system and $\hat{A}$ an orthogonal projector.

It follows from the very definition that probability measures on quantum logics, in particular those defined by the formula (1) on Hilbertian quantum logics, satisfy all numerical constraints imposed on Kolmogorovian probability measures: they are nonnegative, normalized, and σ-additive on families of pairwisely disjoint (in the language of quantum logics: pairwisely orthogonal) elements. However, this surely does not mean that Kolmogorovian probability calculus which is based on Boolean σ-algebras is an adequate tool for quantum mechanics.[3] This is particularly well seen in the quantum logic approach, where several theorems were proved showing that various versions of Bell-type inequalities are satisfied by probability measures defined on a quantum logic iff this logic is a Boolean algebra (see, e.g., Santos, 1986; Pulmannová and Majernik, 1992; Beltrametti and Mączyński, 1993a, b).

Apart from these numerical and "structural" differences between classical and quantum probabilities, there is one more difference mentioned already in the Introduction: quantum random events are not subsets of the space of elementary events, but mathematical objects of another kind. This does not allow one to treat quantum random events as subsets of a phase space of a physical system and probably is a source of difficulties in representing quantum mechanics on a phase space.

In the quantum logic approach, states of any physical system are represented by probability measures on a logic and they form a convex set whose extreme points represent pure states of a system. In the case of the phase space description of classical statistical systems these extreme points are Dirac measures concentrated on one-point subsets of a phase space, so they are usually identified with points of a phase space of a system. Random events are subsets of a phase space and each random event is in an obvious way defined by a property of the physical system: it consists of these pure states for which the given property holds. This allows one to interpret set-theoretic unions and intersections of random events as being generated by disjunctions and conjunctions of propositions about the studied physical system in full accordance with the spirit of Kolmogorovian probability theory.

This is no longer true for quantum systems: properties of a quantum system, represented by elements of a logic, are not subsets of sets of pure states. In the next section we shall see, however, that there is a possibility of representing elements of logics even of "genuine" quantum systems by *fuzzy* subsets of sets of pure states. This not only allows us to make quantum probability calculus more similar to the classical, Kolmogorovian, one, but also could be a first step toward constructing a phase-space representation of quantum mechanics free from the well-known difficulties connected with the appearance of negative probabilities.

[3] Ballentine's (1986) conviction that he has "refuted any and all claims that 'classical' probability theory is not valid in quantum mechanics" seems to be based on the superficial analysis in which he took into account neither Bell-type inequalities nor the difference of structures on which classical and quantum probability measures are defined.

3. FUZZY SET MODELS OF QUANTUM LOGICS

Let $\mathcal{U} \neq \emptyset$ be a fixed set called a *universe.* According to Zadeh (1965), a *fuzzy set A* in $\mathcal{U}$ is defined by its *membership function* $\mu_A: \mathcal{U} \to [0, 1]$ in such a way that for any $x \in \mathcal{U}$ the number $\mu_A(x) \in [0, 1]$ represents the *degree of membership* of x to the fuzzy set A. Giles (1976) noticed that fuzzy sets arise naturally as a result of application of the infinite-valued Łukasiewicz logic to evaluate the truth-value of a proposition "x belongs to A", i.e., that relations of fuzzy sets to the infinite-valued logic are the same as relations of traditional sets (called *crisp sets* in the "fuzzy" literature) to the classical bivalent logic. Many authors identify fuzzy sets with their membership functions and write $A(x)$ instead of $\mu_A(x)$. This convention is adopted throughout the rest of this paper.

It was proved by the author (Pykacz, 1994) [see also Mesiar (1994), Dvurečenskij (1996), and Pykacz (1997a,b) for further generalizations] that any quantum logic L with an ordering set of probability measures S can be isomorphically represented as a family $\mathcal{L}(S)$ of fuzzy subsets of S such that:

- (f.1) $\mathcal{L}(S)$ contains the empty set.
- (f.2) $\mathcal{L}(S)$ is closed with respect to the *standard fuzzy set complementation* $A' = 1 - A$.
- (f.3) $\mathcal{L}(S)$ is closed with respect to Giles unions of pairwise weakly disjoint sets.
- (f.4) The empty set is the only set in $\mathcal{L}(S)$ which is weakly disjoint with itself.

where, for any two fuzzy sets A,B, their *Giles union* $A \sqcup B$ and *Giles intersection* $A \sqcap B$ are defined, respectively, by the formulas

$$(A \sqcup B)(x) = \min[A(x) + B(x), 1] \tag{2}$$

$$(A \sqcap B)(x) = \max[A(x) + B(x) - 1, 0] \tag{3}$$

the *weak disjointness* of two fuzzy sets means that their Giles intersection is the empty set, and the isomorphism between L and $\mathcal{L}(S)$ is given by the formula

$$L \ni a \leftrightarrow A \in \mathcal{L}(S), \qquad A(s) = s(a) \qquad \text{for all} \quad s \in S \tag{4}$$

Conversely, it was shown in Pykacz (1994) that any family $\mathcal{L}(\mathcal{U})$ of fuzzy subsets of an arbitrary universe $\mathcal{U}$ which satisfies conditions (f.1)–(f.4) is a quantum logic in the traditional sense partially ordered by the fuzzy set inclusion: $A \subseteq B$ iff for all $x \in \mathcal{U}$ $A(x) \leq B(x)$, and equipped with the standard fuzzy set complementation as orthocomplementation.

Any family of fuzzy sets satisfying conditions (f.1)–(f.4) will be called a *quantum logic of fuzzy sets* or simply a *fuzzy quantum logic.*

Let us note that the requirement that a quantum logic representing properties of a physical, classical or quantum, system should possess an ordering set of probability measures S is, from a physical point of view, quite natural: Probability measures represent states of a physical system. The only way to establish experimentally an order relation between properties of a physical system is to perform a number of experiments on a system when it is in various states, which means that the set of states has to be ordering. Moreover, if the set of states S were not ordering, one could divide an original logic into suitable equivalence classes in order to make S ordering on the "new" logic. Let us also note that two standard examples of quantum logics—Boolean algebra $\mathcal{B}(\Gamma)$ of Borel subsets of a phase space Γ in classical statistical mechanics and Hilbertian quantum logic $L(\mathcal{H})$ in quantum mechanics—actually do possess ordering sets of probability measures.

The above-described representation of elements of an abstract quantum logic L by family of fuzzy subsets of the set of states S allows us to compare more easily logics of quantum systems and logics of classical statistical systems. Both similarities and characteristic differences between these two kinds of logics are particularly well seen when we restrict ourselves to the set P consisting of pure states only. In both cases, each property $a \in L$ of a physical system Σ defines, by the formula (4), a subset $A \subseteq P$ *consisting of pure states in which the system* Σ *has the property* a (in other words, the set A is defined by the predicate "has the property a"). In the case of classical statistical systems all subsets of P defined in this way are necessarily traditional crisp sets since pure states in classical mechanics are dispersion-free: $A(s) = s(a) \in \{0,1\}$, which expresses the fact that a classical system in a pure state either surely has or surely does not have any if its properties. Therefore, the membership function of the set $A \subseteq P$ is, in this case, a characteristic function and the set A is crisp.

This is no longer the case in quantum mechanics, since here even pure states are in general dispersive, so the set $A \subseteq P$ defined in the way described above is in general a genuine fuzzy, i.e., noncrisp set. Of course now the sentence "the system Σ has the property a" belongs to the domain of infinite-valued logic and as its truth value $A(s) = s(a)$ it can assume any number from the unit interval. Equivalently, and in accordance with the very spirit of the fuzzy set theory, we can say that even when a quantum system is in a pure state, it "has" any of its properties to the degree represented numerically by a number from the interval [0,1].

Nevertheless, if we assume that properties of a physical system form a quantum logic, in both cases the family $\mathcal{L}(P)$ consisting of all fuzzy subsets of P defined in the above-described manner obviously has to satisfy conditions (f.1)–(f.4). In the phase space description of a classical statistical system $\mathcal{L}(P)$ can be identified with a Boolean σ-algebra $\mathcal{B}(\Gamma)$ of Borel subsets of a phase space Γ since it is believed that any such subset represents a property of a classical system. In the Hilbert space description of a quantum system $\mathscr{L}(P)$ can be identified with a family of fuzzy subsets $\{A\}$ of the unit sphere $S^1(\mathscr{H})$ in a Hilbert space $\mathscr{H}$ associated with a system. In this case the fuzzy sets $A \subseteq S^1(\mathscr{H})$ which form the quantum logic $\mathscr{L}(P)$ are defined by the formula

$$A(\Psi) = \langle \hat{A}\Psi, \Psi \rangle \tag{5}$$

where $\Psi \in S^1(\mathscr{H})$ is a unit vector and $\hat{A}$ is an orthogonal projection in $\mathscr{H}$. However, in general it should also be possible to obtain a phase space description of a quantum system by representing Hilbertian quantum logic by a suitable family of fuzzy subsets of Γ instead of representing it by a family of fuzzy subsets of $S^1(\mathscr{H})$.

4. QUANTUM LOGICS OF FUZZY SETS AS A POSSIBLE BASIS FOR QUANTUM PROBABILITY CALCULUS

Giles union and intersection and the standard fuzzy set complementation coincide with traditional, set-theoretic operations when fuzzy sets are replaced by crisp sets, and in this case weak disjointness coincides with the ordinary disjointness of crisp sets. Therefore, the conditions (f.1)–(f.4) that define a fuzzy quantum logic show remarkable similarity to the conditions that define Boolean σ-algebras of events in the Kolmogorovian probability theory. The difference between the condition (f.3) and Kolmogorovian requirement that a σ-algebra of events should be closed with respect to countable unions of arbitrary, not only pairwise disjoint, sets seems to be unimportant since this requirement of Kolmogorov is superfluous: Probability measures are assumed to be σ-additive on pairwise disjoint, not arbitrary sequences of sets and it is possible to construct reasonable "classical" probability theory with this requirement being suitably modified [for a detailed discussion of this problem see Fine (1973)]. Since the condition (f.4) in the domain of crisp sets is trivially satisfied (the empty set is the unique crisp set which is disjoint with itself), we infer that a notion of a fuzzy quantum logic is in a sense a "minimal" generalization of the notion of an algebra of random events to a family of fuzzy sets endowed with Giles connectives, which allows one to build a reasonable probability calculus.

It should be mentioned that it is possible to build a fuzzy probability theory using, instead of Giles operations, other operations chosen from the vast family of fuzzy unions and intersections. This was actually done in the number of papers (see, e.g., Zadeh, 1968; Klement *et al.*, 1981; Piasecki, 1985; Mesiar, 1992) in which a fully-fledged fuzzy probability theory was developed. This theory was also, especially in the papers of the "Slovak School" too numerous to be listed here [see, e.g., Riečan (1992) and references listed therein, or the bibliographical essay by Cattaneo *et al.* (n.d.)], used to investigate various typically "physical" notions. However, in the majority of these papers their authors use the original Zadeh (1965) operations, which cannot be used to build fuzzy set models of quantum logics, since they, when combined with the standard fuzzy set complementation, do not satisfy the excluded middle law and the law of contradiction for any genuine fuzzy set (Pykacz, 1994; Mesiar, 1994). Therefore, it seems that only Giles operations, or other operations isomorphic to them pointwisely defined on fuzzy sets by nilpotent triangular (co)norms, can be sensibly used to build fuzzy set models of quantum logics (Mesiar, 1994; Pykacz, 1997a) and therefore also to construct fuzzy set models of quantum probability spaces.

One problem still deserves explanation: There do exist "genuine" (i.e., non-Boolean) quantum logics consisting of traditional, crisp sets, so-called *concrete logics* (see, e.g., Pták and Pulmannová, 1991). They are defined as families of crisp sets that satisfy crisp counterparts of the conditions (f.1)–(f.3) [as we already noticed, the crisp counterpart of the condition (f.4) is always satisfied]. The natural question arises of why we should bother at all about fuzzy sets, and why not be satisfied with concrete logics in attempts to make quantum probability calculus more similar to the classical, Kolmogorovian, one? The answer can be found on the pp. 23 and 24 of Pták and Pulmannová (1991): Every concrete logic has a (strongly) ordering set of dispersion-free states, but it follows from Gleason's theorem that a Hilbertian quantum logic $L(\mathcal{H})$ with $dim(\mathcal{H}) \geq 3$ does not possess any dispersion-free state. Therefore, we cannot expect that concrete quantum logics could be of any value for studying properties of really existing quantum systems which are representable in Hilbert spaces.

After replacing abstract quantum logics that appear in the foundations of quantum probability calculus by families of fuzzy sets which satisfy conditions (f.1)–(f.4) one obtains, at the price of allowing fuzzy sets to come into play, a perfect parallelism between Kolmogorovian probability calculus applied to classical statistical systems and quantum probability calculus applied to quantum systems: In both cases random events are represented by subsets of sets of pure states of physical systems and they are defined by properties of these systems. Conjunctions and disjunctions of properties of physical systems define intersections and unions of respective subsets. However, it should be stressed that, contrary to the situation encountered in classical statistical physics, in quantum physics the results of these operations do not always belong to a (fuzzy) quantum logic even if this logic is a lattice [see Pykacz (1994, 1997b) for a detailed discussion of this problem]. Therefore, the usage of joins and meets in order to construct "compound" quantum random events—a common practice in quantum probability—instead of Giles unions and intersections can be a source of serious mistakes.

As already mentioned, it should be possible in general to represent properties of a quantum system by a family of fuzzy subsets of a phase space Γ instead of fuzzy subsets of a unit sphere $S^1(\mathcal{H})$ of a Hilbert space $\mathcal{H}$, obtaining in this way a phase-space representation of a quantum system. Such a representation could be obtained in suitable cases, e.g., by mapping points $\Psi \in S^1(\mathcal{H})$ onto points $(\langle p\rangle_\Psi, \langle q\rangle_\Psi) \in \Gamma$ with $\langle p\rangle_\Psi$, $\langle q\rangle_\Psi$ being, respectively, mean values of the momentum and the position operators in a state Ψ. A value $A(\langle p\rangle_\Psi, \langle q\rangle_\Psi)$ of a membership function of a fuzzy subset $A \subset \Gamma$ that represents a property a should in this case be given by the formula (5), i.e.,

$$A(\langle p\rangle_\Psi, \langle q\rangle_\Psi) = \langle \hat{A}\Psi, \Psi\rangle \tag{6}$$

where $\hat{A} \in L(\mathcal{H})$ is an orthogonal projection representing the property a in the Hilbertian quantum logic $L(\mathcal{H})$. Of course, numerical values of all probability measures defined on a logic of properties of a quantum system have to remain the same since it makes no difference whether properties of a system are represented by closed subspaces of a Hilbert space, orthogonal projections onto these subspaces, fuzzy subsets of the unit sphere in a Hilbert space, or suitably defined fuzzy subsets of a phase space.

Therefore, the above-sketched phase-space representation of quantum systems should be free of such counterintuitive ingredients like negative probabilities which have plagued phase-space representations of quantum mechanics from the very birth of this idea. In the author's opinion the necessity of working with σ-orthomodular posets of fuzzy subsets instead of Boolean σ-algebras of crisp subsets of a phase space is not too high price to be paid for this.

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